The Moore-Penrose Inverse Over a Commutative Ring

R. B. Bapat
Indian Statistical Institute
New Delhi, 110016, India

and

Donald W. Robinson
Department of Mathematics
Brigham Young University
Provo, Utah 84602

Submitted by Richard A. Brualdi

ABSTRACT

Let $\mathbf{R}$ be a commutative ring with 1 and with an involution $a \to \bar{a}$, and let $\mathbf{M_R}$ be the category of finite matrices over $\mathbf{R}$ with the involution $(a_{ij}) \to (a_{ij})^* = (\bar{a}_{ji})$. A matrix $A: m \to n$ in $\mathbf{M_R}$ of determinantal rank r such that

$$u(A) = \sum_{\alpha \in Q_{r,m}} \sum_{\beta \in Q_{r,n}} \det A_{\alpha\beta} \overline{\det A_{\alpha\beta}}$$

has a Moore-Penrose inverse $u(A)^\dagger$ in $\mathbf{R}$ is said to be Moore invertible with Moore idempotent $u(A)u(A)^\dagger$ if $u(A)u(A)^\dagger A = A$. For every matrix A of $\mathbf{M_R}$, A has a Moore-Penrose inverse with respect to * if and only if A is the sum of Moore invertible matrices whose Moore idempotents are pairwise orthogonal.

INTRODUCTION

It is well known that the inverse of a square complex matrix with nonzero determinant may be expressed in terms of the adjoint of the matrix. In 1920, E. H. Moore extended this classical notion to provide a formula for what is now termed the Moore-Penrose inverse of an arbitrary complex matrix. (See

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[13].) Several others have also provided representations of the elements of this generalized inverse in terms of rational functions of certain determinants of submatrices of the given matrix. (See, for example, [1], [6], [7], [8], [9], [10], [4], and [5].)

These representations have been used to characterize the existence of the Moore-Penrose inverse in specific categories. In particular, it is known that a matrix A of rank r over a field admits a Moore-Penrose inverse with respect to the involution of transpose if and only if the sum of the squares of the $r \times r$ minors of A is a unit. (See, for example, [7, p. 109].) Recently, this characterization has been extended to matrices over an integral domain by R. B. Bapat, K. P. S. B. Rao, and K. M. Prasad in [2]. Their techniques require the embedding of the domain in a field, which in turn permits the use of the full rank factorization theorem for matrices.

The purpose of the present paper is to provide an extension of these results to matrices over any commutative ring with 1. Specifically, we characterize those matrices which possess Moore-Penrose inverses in this category.

Throughout this paper $\mathbf{R}$ is taken to be a commutative ring with 1 and with an involution $\bar{\ }$. That is, $\bar{\ }$ is a mapping $a \to \bar{a}$ on $\mathbf{R}$ such that for every a and b in $\mathbf{R}$,

$$\overline{a+b} = \bar{a} + \bar{b}, \qquad \overline{ab} = \bar{b}\bar{a}, \qquad \bar{\bar{a}} = a.$$

In particular, $\bar{0} = 0$ and $\bar{1} = 1$. Since $\mathbf{R}$ is commutative, one possible such involution is the identity mapping.

Also, throughout this paper $\mathbf{M_R}$ is understood to be the category of finite matrices over $\mathbf{R}$ with the involution $(a_{ij}) \to (a_{ij})^* = (\bar{a}_{ji})$. In particular,

$$(A+B)^* = A^* + B^*, \qquad (AB)^* = B^*A^*, \quad \text{and} \quad A^{**} = A$$

whenever the compatibility conditions on the sizes of the matrices are satisfied.

A matrix A in $\mathbf{M_R}$ is said to have a Moore-Penrose inverse in $\mathbf{M_R}$ provided that there is a matrix $A^\dagger$ in $\mathbf{M_R}$ such that

$$AA^\dagger A = A, \qquad A^\dagger AA^\dagger = A^\dagger, \qquad (AA^\dagger)^* = AA^\dagger, \qquad (A^\dagger A)^* = A^\dagger A.$$

If such an $A^\dagger$ exists, then it is unique, and is called the Moore-Penrose inverse of A. (See, for example, [14, p. 132].)

Now, let A be an $m \times n$ matrix over $\mathbf{R}$ of determinantal rank r. If A possesses a Moore-Penrose inverse in $\mathbf{M}_{\mathbf{R}}$, then it is shown below that

$$u(A) = \sum_{\alpha \in Q_{r,m}} \sum_{\beta \in Q_{r,n}} \det A_{\alpha\beta} \overline{\det A_{\alpha\beta}}$$

has a Moore-Penrose inverse in $\mathbf{R}$, where $Q_{r,m}$ is the totality of lists $\alpha = (\alpha(1), \dots, \alpha(r))$ of integers with $1 \leqslant \alpha(1) < \cdots < \alpha(r) \leqslant m$, and $\det A_{\alpha\beta}$ is the determinant of the submatrix of A determined by the rows $\alpha(1), \dots, \alpha(r)$ and the columns $\beta(1), \dots, \beta(r)$ with $\alpha = (\alpha(1), \dots, \alpha(r)) \in Q_{r,m}$ and $\beta = (\beta(1), \dots, \beta(r)) \in Q_{r,n}$. If $r = 0$, then by convention, $u(A) = 1$.

On the other hand, it is also shown that if $u(A)^{\dagger}$ exists and $u(A)u(A)^{\dagger}A = A$, then A has the $n \times m$ Moore-Penrose inverse $A^{\dagger} = (g_{ij}(A))$ in $\mathbf{M}_{\mathbf{R}}$ with

$$g_{ij}(A) = [u(A)]^{\dagger} \sum_{\substack{\alpha \in Q_{r,m} \\ j \in \alpha}} \sum_{\substack{\beta \in Q_{r,n} \\ i \in \beta}} \overline{\det A_{\alpha\beta}} \cdot \left(\text{cofactor of } a_{ji} \text{ in } A_{\alpha\beta}\right),$$

where, for example, the first summation is taken over those lists $\alpha = (\alpha(1), \dots, \alpha(r))$ in $Q_{r,m}$ for which $j = \alpha(t)$ for some index t. This formula for $A^{\dagger}$ generalizes the familiar expression for the inverse of an invertible matrix. Indeed, since $\mathbf{R}$ is commutative, then in this case $m = n = r$, $Q_{r,n}$ consists of the singleton $(1, \dots, n)$, and $\det A$ is invertible in $\mathbf{R}$; thus, $u(A) = \det A \overline{\det A}$ is invertible, and the cofactor of a_{ji} in A is the (i, j) element of the classical adjoint of A. Consequently,

$$A^{\dagger} = \left(\det A \overline{\det A}\right)^{-1} \left(\overline{\det A} \operatorname{adj} A\right) = (\det A)^{-1} \operatorname{adj} A = A^{-1}.$$

1. PRELIMINARIES

In this section four lemmas are provided that are used in the proof of the main result of this paper.

LEMMA 1. *Let*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : s + r \to s + r$$

in $\mathbf{M}_{\mathbf{R}}$ *be of determinantal rank* $\leqslant r$. *Then*

$$\det\begin{pmatrix} 0 & B \\ C & D \end{pmatrix} = \det(-A)\det D.$$

Proof. Let $\mathscr{A}_i$ and $\mathscr{B}_{s-i}$ denote the matrices consisting of the first i rows of A and the last $s-i$ rows of B, respectively. Let

$$M_0 = \begin{pmatrix} 0 & B \\ C & D \end{pmatrix}, \qquad M_s = \begin{pmatrix} -A & 0 \\ C & D \end{pmatrix},$$

and for $0 < i < s$ let

$$M_i = \begin{pmatrix} -\mathscr{A}_i & 0 \\ 0 & \mathscr{B}_{s-i} \\ C & D \end{pmatrix}.$$

Now, M_{i-1} and M_i are identical except for their ith rows. Specifically, if a_i and b_i denote the ith rows of A and B, respectively, then the ith rows of M_{i-1} and M_i are, respectively, $(0, b_i)$ and $(-a_i, 0)$. Since $(0, b_i) = (-a_i, 0) + (a_i, b_i)$, then

$$\det M_{i-1} = \det M_i + \det\begin{pmatrix} -\mathscr{A}_{i-1} & 0 \\ a_i & b_i \\ 0 & \mathscr{B}_{s-i} \\ C & D \end{pmatrix}.$$

But, since the $r+1$ rows $i, s+1, \dots, s+r$ of this last expression are rows of the given matrix, which by hypothesis is of determinantal rank at most r, then the Laplace expansion of this determinant by these rows gives a value of zero. (See, for example, [11, p. 14].) That is, $\det M_{i-1} = \det M_i$ for $i = 1, \dots, s$. Consequently,

$$\det\begin{pmatrix} 0 & B \\ C & D \end{pmatrix} = \det M_0 = \det M_s = \det\begin{pmatrix} -A & 0 \\ C & D \end{pmatrix} = \det(-A)\det D. \quad \blacksquare$$

COROLLARY 1.1. *Let the conditions be as in Lemma* 1. *If* $s = r$, *then* $\det B \det C = \det A \det D$.

Proof. By use of Lemma 1 with $s = r$,

$$\det B \det C = \det\begin{pmatrix} B & 0 \\ D & C \end{pmatrix} = (-1)^r \det\begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$$

$$= (-1)^r \det(-A) \det D = (-1)^r(-1)^r \det A \det D$$

$$= \det A \det D. \qquad \blacksquare$$

COROLLARY 1.2. *Let the conditions be as in Lemma* 1 *with* $s = r - 1$. *Let* B_j *be the* $(r-1) \times (r-1)$ *matrix obtained from* B *by the deletion of the* j*th column, and let* C_i *be the* $(r-1) \times (r-1)$ *matrix obtained from* C *by the deletion of the* i*th row. Then*

$$\sum_{i=1}^{r} \sum_{j=1}^{r} (-1)^{i+j} (\det C_i) d_{ij} (\det B_j) = \det A \det D.$$

Proof. If $r = 1$, then the result is valid by the agreement that the determinant of the empty matrix is 1. Thus, let $r > 1$. The Laplace expansion of

$$\det\begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$$

by the first $r - 1$ rows gives

$$(-1)^{1+2+\cdots+(r-1)} \sum_{j=1}^{r} (-1)^{r+(r+1)+\cdots+[r+(r-1)]-[r+(j-1)]} \det B_j \det(C, d_j),$$

where d_j is the jth column of D. (See, for example, [11, p. 14].) Consequently,

$$\det A \det D = (-1)^{r-1} \det(-A) \det D = (-1)^{r-1} \det\begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$$

$$= (-1)^{(r-1)+r(r-1)/2+(r-1)^2+r(r+1)/2} \sum_{j=1}^{r} (-1)^j \det B_j \det(C, d_j)$$

$$= (-1)^r \sum_{j=1}^{r} (-1)^j \det B_j \sum_{i=1}^{r} (-1)^{i+r} d_{ij} \det C_i$$

$$= \sum_{i=1}^{r} \sum_{j=1}^{r} (-1)^{i+j} (\det B_j) d_{ij} (\det C_i). \qquad \blacksquare$$

The authors extend appreciation to Wayne W. Barrett for his assistance in the formulation of Lemma 1.

LEMMA 2. *Let A in $\mathbf{M}_{\mathbf{R}}$ be of determinantal rank* 1, *and let* $a = \operatorname{tr} AA^*$. *Then A has a Moore-Penrose inverse in $\mathbf{M}_{\mathbf{R}}$ if and only if a has a Moore-Penrose inverse in* $\mathbf{R}$ *and* $aa^\dagger A = A$. *In this case,* $A^\dagger = a^\dagger A^*$, $(AA^*)^\dagger = A^{\dagger *} A^\dagger$, *and* $a^\dagger = \operatorname{tr}(AA^*)^\dagger$.

Proof. Since $\mathbf{R}$ is commutative and $A = (a_{ij}) : m \to n$ is of rank 1, then for all possible subscripts, $a_{ik}a_{hj} = a_{ij}a_{hk}$. Therefore,

$$(AA^*A)_{ij} = \sum_{k=1}^{n} \sum_{h=1}^{m} a_{ik}\bar{a}_{hk}a_{hj} = \sum_{h=1}^{m} \sum_{k=1}^{n} a_{hk}\bar{a}_{hk}a_{ij} = (\operatorname{tr} AA^*)a_{ij}.$$

That is, $AA^*A = aA$, and $(AA^*)^2 = aAA^*$.

Suppose first that $a^\dagger$ exists, and that $aa^\dagger A = A$. Since AA^* is symmetric with respect to $*$, then tr AA^* is symmetric with respect to $\bar{\ }$; that is, $\bar{a} = a$ and hence $\overline{a^\dagger} = a^\dagger$. It now follows that $a^\dagger A^*$ is the Moore-Penrose inverse of A. Indeed, $A(a^\dagger A^*) = a^\dagger AA^*$ and $(a^\dagger A^*)A = a^\dagger A^*A$ are both symmetric with respect to $*$; by hypothesis, $A(a^\dagger A^*)A = a^\dagger AA^*A = a^\dagger aA = aa^\dagger A = A$; and $(a^\dagger A^*)A(a^\dagger A^*) = a^{\dagger 2}A^*AA^* = a^{\dagger 2}\bar{a}A^* = a^\dagger A^*$. Consequently, $A^\dagger$ exists, and $A^\dagger = a^\dagger A^*$.

Conversely, suppose that $A^\dagger$ exists in $\mathbf{M}_{\mathbf{R}}$. By [15, Lemma 1.2], $(AA^*)^\dagger = A^{\dagger *}A^\dagger$ exists, which by [15, Lemma 1.1] is also the group inverse of AA^*. Therefore,

$$(AA^*)^\dagger = (AA^*)^2\left[(AA^*)^\dagger\right]^3 = aAA^*\left[(AA^*)^\dagger\right]^3 = a\left[(AA^*)^\dagger\right]^2.$$

Since $\bar{a} = a$, again by [15, Lemma 1.1],

$$AA^* = (AA^*)^{\dagger\dagger} = (aI_m)(AA^*)^\dagger(aI_m)^* = a^2(AA^*)^\dagger.$$

Thus, $a = \operatorname{tr} AA^* = a^2 \operatorname{tr}(AA^*)^\dagger$. By [15, Lemma 1.1], $a^\dagger$ exists; indeed, with $b = \operatorname{tr}(AA^*)^\dagger$, since $\bar{b} = b$, then $a^\dagger = b^2 a$.

Furthermore, since $(AA^*)^\dagger = A^{\dagger *}A^\dagger = (A^{\dagger *})(A^{\dagger *})^*$, the same argument provides $(AA^*)^\dagger = b^2 AA^*$ and $b = \operatorname{tr}(AA^*)^\dagger = b^2 a$. That is, $a^\dagger = b$. Consequently, $a^\dagger = \operatorname{tr}(AA^*)^\dagger$,

$$A^\dagger = A^*(AA^*)^\dagger = A^*(a^{\dagger 2}AA^*) = a^{\dagger 2}A^*AA^* = a^{\dagger 2}(aA)^* = a^{\dagger 2}aA^* = a^\dagger A^*,$$

and

$$A = AA^{\dagger}A = A(a^{\dagger}A^{*})A = a^{\dagger}AA^{*}A = a^{\dagger}aA = aa^{\dagger}A. \qquad \blacksquare$$

The results of Lemma 2 are now applied to the r-compound $C_r(A)$ of a matrix A of rank r. (See, for example, [11, p. 16].)

COROLLARY 2. *Let A in $\mathbf{M_R}$ be of determinantal rank r, and let $u(A)$ be given as above. If $A^{\dagger}$ exists in $\mathbf{M_R}$, then $u(A)^{\dagger}$ exists in $\mathbf{R}$, $u(A)u(A)^{\dagger}C_r(A) = C_r(A)$, $\operatorname{rank}[u(A)u(A)^{\dagger}A] = r$, and $u(u(A)u(A)^{\dagger}A) = u(A)$.*

Proof. By the properties of the r-compound, $[C_r(A)]^{\dagger} = C_r(A^{\dagger})$. (See, for example, [11, p. 17].) By Corollary 1.1, $C_r(A)$ is of determinantal rank 1. [By convention, $C_0(0) = (1) : 1 \to 1$.] By an application of Lemma 2 to the matrix $C_r(A)$, $u(A) = \operatorname{tr} C_r(A)[C_r(A)]^{*}$ has a Moore-Penrose inverse $u(A)^{\dagger}$ in $\mathbf{R}$ with respect to $\bar{\ }$, and $u(A)u(A)^{\dagger}C_r(A) = C_r(A)$. In particular, since $e(A) = u(A)u(A)^{\dagger}$ is idempotent, $C_r(e(A)A) = e(A)^{r}C_r(A) = e(A)C_r(A) = C_r(A)$. Since $C_r(A) \neq 0$, then $C_r(e(A)A) \neq 0$ and $\operatorname{rank}[e(A)A] \geqslant r$. Clearly, $\operatorname{rank}[e(A)A] \leqslant \operatorname{rank} A = r$. Thus, $\operatorname{rank}[e(A)A] = r$ and $u(e(A)A) = u(A)$. $\blacksquare$

Now, let A in $\mathbf{M_R}$ be of determinantal rank r, and suppose that $A^{\dagger}$ exists. Since $u(A)u(A)^{\dagger}C_r(A) = C_r(A) \neq 0$, if $\mathbf{R}$ is an integral domain, then $u(A)u(A)^{\dagger} = 1$ and $u(A)$ is invertible in $\mathbf{R}$. This result is known. (See, for example, [2].) It is also clear that the same conclusion follows whenever an $r \times r$ minor of A, hence an entry of $C_r(A)$, is not a zero divisor in $\mathbf{R}$. Likewise, if the only symmetric idempotents of $\mathbf{R}$ are the trivial 0 and 1, then again $u(A)u(A)^{\dagger} = 1$ and $u(A)$ is invertible. (For related results see also [3], [12, Application 2], and [16].)

On the other hand, if $u(A)$ is invertible, then $A^{\dagger}$ exists. Indeed, a formula due to E. H. Moore, which provides $A^{\dagger}$ explicitly in terms of the $r \times r$ minors of A, is shown below to be valid in $\mathbf{M_R}$. More generally, a slight modification of this formula is now shown to provide an expression for $A^{\dagger}$ whenever $u(A)u(A)^{\dagger}A = A$.

Specifically, for $\alpha = (\alpha(1), \ldots, (\alpha(r)) \in Q_{r,m}$ and $1 \leqslant j \leqslant m$, let $j \in \alpha$ mean that $j = \alpha(t)$ for some index t; in this case, this unique index is denoted by $t = j(\alpha)$. Similarly, for $i \in \beta = (\beta(1), \ldots, \beta(r)) \in Q_{r,n}$, let $i(\beta)$ be the unique index s such that $\beta(s) = i$. In particular, if $j \in \alpha$ and $i \in \beta$, then the cofactor of the element a_{ji} in the submatrix $A_{\alpha\beta}$ of $A : m \to n$ is given by

$$(-1)^{j(\alpha)+i(\beta)} \det A_{\alpha_j \beta_i},$$

where α_j is the list of $Q_{r-1,m}$ consisting of the entries of α with j omitted, and β_i is the list of $Q_{r-1,n}$ consisting of the entries of β with i omitted.

Furthermore, for $j \in \alpha$ and $1 \leqslant k \leqslant m$, let $A_{\alpha(j \leftarrow k), \beta}$ be the $r \times r$ matrix that is the same as $A_{\alpha\beta}$ except that the elements $a_{j\beta(s)}$ of row $j(\alpha)$ have been replaced by the corresponding elements $a_{k\beta(s)}$.

LEMMA 3. *Let* $A = (a_{ij}) : m \to n$ *in* $\mathbf{M_R}$ *be of determinant rank* r, $\alpha = (\alpha(1), \dots, \alpha(r)) \in Q_{r,m}$, $\beta = (\beta(1), \dots, \beta(r)) \in Q_{r,n}$, $j \in \alpha$, $1 \leqslant k \leqslant m$, *and* $1 \leqslant h \leqslant n$. *Then*

(1) $\det A_{\alpha(j \leftarrow k), \beta} = \sum_{i \in \beta} (-1)^{j(\alpha)+i(\beta)} a_{ki} \det A_{\alpha_j \beta_i}$,
(2) $a_{kh} \det A_{\alpha\beta} = \sum_{j \in \alpha} a_{jh} \det A_{\alpha(j \leftarrow k), \beta}$.

Proof. (1): By determinant expansion via a row, and by a change of summation index via $i = \beta(s)$, $i(\beta) = s$,

$$\begin{aligned} \det A_{\alpha(j \leftarrow k), \beta} &= \sum_{s=1}^{r} (-1)^{j(\alpha)+s} a_{k\beta(s)} \det A_{\alpha_j \beta_{\beta(s)}} \\ &= \sum_{i \in \beta} (-1)^{j(\alpha)+i(\beta)} a_{ki} \det A_{\alpha_j \beta_i}. \end{aligned}$$

(2): Let $A_{(\alpha, k),(\beta, h)}$ be the $(r+1) \times (r+1)$ matrix determined by rows $(\alpha(1), \dots, \alpha(r), k)$ and columns $(\beta(1), \dots, \beta(r), h)$ of A. By duplication either of rows or of columns [for example, $k = \alpha(t)$ for some t or $h = \beta(s)$ for some s] or by the hypothesis that the determinants of all $(r+1) \times (r+1)$ submatrices of A are zero, it follows that $\det A_{(\alpha, k),(\beta, h)} = 0$. An expansion of this determinant by the last column gives

$$0 = \sum_{t=1}^{r} (-1)^{t+(r+1)} a_{\alpha(t), h} \det A_{(\alpha_{\alpha(t)}, k), \beta} + (-1)^{(r+1)+(r+1)} a_{kh} \det A_{\alpha\beta}.$$

Thus, by a change of summation index via $j = \alpha(t)$, $j(\alpha) = t$,

$$\begin{aligned} a_{kh} \det A_{\alpha\beta} &= \sum_{j \in \alpha} (-1)^{j(\alpha)+r} a_{jh} \det A_{(\alpha_j, k), \beta} \\ &= \sum_{j \in \alpha} (-1)^{j(\alpha)+r} a_{jh} (-1)^{r-j(\alpha)} \det A_{\alpha(j \leftarrow k), \beta} \\ &= \sum_{j \in \alpha} a_{jh} \det A_{\alpha(j \leftarrow k), \beta}. \end{aligned}$$

■

LEMMA 4. *Let* $A : m \to n$ *in* $\mathbf{M_R}$ *be of determinantal rank* r, *let*

$$u(A) = \sum_{\alpha \in Q_{r,m}} \sum_{\beta \in Q_{r,n}} \det A_{\alpha\beta} \overline{\det A_{\alpha\beta}}$$

have a Moore-Penrose inverse $u(A)^\dagger$ *in* $\mathbf{R}$, *and let*

$$g_{ij}(A) = u(A)^\dagger \sum_{\substack{\alpha \in Q_{r,m} \\ j \in \alpha}} \sum_{\substack{\beta \in Q_{r,n} \\ i \in \beta}} (-1)^{j(\alpha)+i(\beta)} \overline{\det A_{\alpha\beta}} \det A_{\alpha_j \beta_i}.$$

Then $G(A) = (g_{ij}(A)) : n \to m$ *is the Moore-Penrose inverse of* A *if and only if* $u(A)u(A)^\dagger A = A$.

Proof. Let the conditions be as in the statement of the lemma. It is now shown that $G(A) = (g_{ij}(A)) : n \to m$ satisfies the equations which define the Moore-Penrose inverse of A if and only if $u(A)u(A)^\dagger A = A$. For convenience, $\Sigma_{\alpha \in Q_{r,m}}$ is abbreviated as simply Σ_α, and $G(A)$ as G.

First, by an interchange of the order of summation, and by part (1) of Lemma 3,

$$\begin{aligned}
(AG)_{kj} &= \sum_{i=1}^{n} a_{ki} g_{ij} \\
&= \sum_{i=1}^{n} a_{ki} \left(u(A)^\dagger \sum_{\substack{\alpha \\ j \in \alpha}} \sum_{\substack{\beta \\ i \in \beta}} (-1)^{j(\alpha)+i(\beta)} \overline{\det A_{\alpha\beta}} \det A_{\alpha_j \beta_i} \right) \\
&= u(A)^\dagger \sum_{\substack{\alpha \\ j \in \alpha}} \sum_{\beta} \overline{\det A_{\alpha\beta}} \sum_{\substack{i \\ i \in \beta}} (-1)^{j(\alpha)+i(\beta)} a_{ki} \det A_{\alpha_j \beta_i} \\
&= u(A)^\dagger \sum_{\substack{\alpha \\ j \in \alpha}} \sum_{\beta} \overline{\det A_{\alpha\beta}} \det A_{\alpha(j \leftarrow k), \beta}.
\end{aligned}$$

Now, if $j = k$, then $\det A_{\alpha(j \leftarrow k), \beta} = \det A_{\alpha\beta}$ and $(AG)_{jj} = u(A)u(A)^\dagger = \overline{(AG)_{jj}}$. Thus, suppose that $j \neq k$. Since, by duplication of rows, $k \in \alpha$ implies $\det A_{\alpha(j \leftarrow k), \beta} = 0$, then

$$(AG)_{kj} = u(A)^\dagger \sum_{\substack{\alpha \\ j \in \alpha \\ k \notin \alpha}} \sum_{\beta} \overline{\det A_{\alpha\beta}} \det A_{\alpha(j \leftarrow k), \beta}.$$

For $j \in \alpha$ and $k \notin \alpha$, let α_{jk} be the list of $Q_{r,m}$ obtained from α by the deletion of j and the inclusion of k. In particular, $\alpha \to \alpha_{jk}$ provides a bijection of the lists of $Q_{r,m}$ which contain j but not k to the lists of $Q_{r,m}$ which contain k but not j, and

$$\det A_{\alpha(j \leftarrow k), \beta} = (-1)^{j(\alpha)+k(\alpha_{jk})} \det A_{\alpha_{jk}, \beta}.$$

Therefore, by a change in the index of summation via $\gamma = \underline{\alpha}_{jk}$, $\alpha = \gamma_{kj}$, and by use of the fact that $u(A)^{\dagger}$ is symmetric with respect to $\bar{\ }$,

$$
\begin{aligned}
(\overline{AG})_{kj} &= u(A)^{\dagger} \sum_{\substack{\alpha \\ j \in \alpha \\ k \notin \alpha}} \sum_{\beta} \det A_{\alpha\beta} \overline{\det A_{\alpha(j \leftarrow k), \beta}} \\
&= u(A)^{\dagger} \sum_{\substack{\alpha \\ j \in \alpha \\ k \notin \alpha}} \sum_{\beta} (-1)^{j(\alpha) + k(\alpha_{jk})} \det A_{\alpha\beta} \overline{\det A_{\alpha_{jk}, \beta}} \\
&= u(A)^{\dagger} \sum_{\substack{\gamma \\ k \in \gamma \\ j \notin \gamma}} \sum_{\beta} (-1)^{j(\gamma_{kj}) + k(\gamma)} \det A_{\gamma_{kj}, \beta} \overline{\det A_{\gamma\beta}} \\
&= u(A)^{\dagger} \sum_{\substack{\gamma \\ k \in \gamma \\ j \notin \gamma}} \sum_{\beta} \overline{\det A_{\gamma\beta}} \det A_{\gamma(k \leftarrow j), \beta} = (AG)_{jk}.
\end{aligned}
$$

Consequently, $(AG)^* = AG$.

Similarly, $(GA)^* = GA$ with

$$
(GA)_{ih} = u(A)^{\dagger} \sum_{\alpha} \sum_{\substack{\beta \\ i \in \beta}} \overline{\det A_{\alpha\beta}} \det A_{\alpha, \beta(i \leftarrow h)}.
$$

Next by use of part (2) of Lemma 3,

$$
\begin{aligned}
(AGA)_{ij} &= \sum_{h=1}^{m} (AG)_{ih} a_{hj} \\
&= \sum_{h} \left(u(A)^{\dagger} \sum_{\substack{\alpha \\ h \in \alpha}} \sum_{\beta} \overline{\det A_{\alpha\beta}} \det A_{\alpha(h \leftarrow i), \beta} \right) \cdot a_{hj} \\
&= u(A)^{\dagger} \sum_{\alpha} \sum_{\beta} \overline{\det A_{\alpha\beta}} \sum_{\substack{h \\ h \in \alpha}} a_{hj} \det A_{\alpha(h \leftarrow i), \beta} \\
&= u(A)^{\dagger} \sum_{\alpha} \sum_{\beta} \overline{\det A_{\alpha\beta}}\, a_{ij} \det A_{\alpha\beta} \\
&= u(A)^{\dagger} \left(\sum_{\alpha} \sum_{\beta} \det A_{\alpha\beta} \overline{\det A_{\alpha\beta}} \right) \cdot a_{ij}.
\end{aligned}
$$

That is, $AGA = u(A)u(A)^{\dagger}A$.

Finally, by use of Corollaries 1.1 and 1.2 applied to the matrices

$$\begin{pmatrix} A_{\gamma\beta} & A_{\gamma\delta} \\ A_{\alpha\beta} & A_{\alpha\delta} \end{pmatrix}, \qquad \begin{pmatrix} A_{\gamma_j\beta_i} & A_{\gamma_j\delta} \\ A_{\alpha\beta_i} & A_{\alpha\delta} \end{pmatrix}$$

it follows that

$$\begin{aligned}
(GAG)_{ij} &= \sum_{h=1}^{n} \sum_{k=1}^{m} g_{ik} a_{kh} g_{hj} \\
&= \sum_{h} \sum_{k} \left(u(A)^{\dagger} \sum_{\substack{\alpha \\ k \in \alpha}} \sum_{\substack{\beta \\ i \in \beta}} (-1)^{k(\alpha)+i(\beta)} \overline{\det A_{\alpha\beta}} \det A_{\alpha_k\beta_i} \right) \\
&\quad \times a_{kh} \cdot \left(u(A)^{\dagger} \sum_{\substack{\gamma \\ j \in \gamma}} \sum_{\substack{\delta \\ h \in \delta}} (-1)^{j(\gamma)+h(\delta)} \overline{\det A_{\gamma\delta}} \det A_{\gamma_j\delta_h} \right) \\
&= \left[u(A)^{\dagger}\right]^2 \sum_{\alpha} \sum_{\substack{\beta \\ i \in \beta}} \sum_{\substack{\gamma \\ j \in \gamma}} \sum_{\delta} (-1)^{i(\beta)+j(\gamma)} \overline{\det A_{\alpha\beta}}\ \overline{\det A_{\gamma\delta}} \\
&\quad \times \sum_{h \in \delta} \sum_{k \in \alpha} (-1)^{k(\alpha)+h(\delta)} \det A_{\alpha_k\beta_i} \cdot a_{kh} \cdot \det A_{\gamma_j\delta_h} \\
&= \left[u(A)^{\dagger}\right]^2 \sum_{\alpha} \sum_{\substack{\beta \\ i \in \beta}} \sum_{\substack{\gamma \\ j \in \gamma}} \sum_{\delta} (-1)^{i(\beta)+j(\gamma)} \overline{\det A_{\gamma\beta}}\ \overline{\det A_{\alpha\delta}} \\
&\quad \times \det A_{\gamma_j\beta_i} \det A_{\alpha\delta} \\
&= u(A)^{\dagger} \left(\sum_{\alpha} \sum_{\delta} \det A_{\alpha\delta} \overline{\det A_{\alpha\delta}} \right) \\
&\quad \times u(A)^{\dagger} \sum_{\substack{\beta \\ i \in \beta}} \sum_{\substack{\gamma \\ j \in \gamma}} (-1)^{j(\gamma)+i(\beta)} \overline{\det A_{\gamma\beta}} \det A_{\gamma_j\beta_i} \\
&= u(A)^{\dagger} u(A) \cdot g_{ij} = g_{ij},
\end{aligned}$$

by use of the fact that $u(A)^{\dagger}u(A)u(A)^{\dagger} = u(A)^{\dagger}$. That is, $GAG = G$.

Consequently, since $AGA = u(A)u(A)^{\dagger}A$, $GAG = G$, $(AG)^* = AG$, and $(GA)^* = GA$, then $A^{\dagger} = G$ if and only if $u(A)u(A)^{\dagger}A = A$. ■

2. THE MAIN RESULT

We say that A is Moore invertible if $A^{\dagger} = G(A)$, where $G(A)$ is defined as in Lemma 4. By Corollary 2 and Lemma 4, A is Moore invertible if and only if $u(A)^{\dagger}$ exists and $u(A)u(A)^{\dagger}A = A$. In this case we call $u(A)u(A)^{\dagger}$ the Moore idempotent of A.

The main result of this paper is now established.

THEOREM. *Let* $\mathbf{R}$ *and* $\mathbf{M}_{\mathbf{R}}$ *be given as above. Then* A *in* $\mathbf{M}_{\mathbf{R}}$ *has a Moore-Penrose inverse* $A^{\dagger}$ *in* $\mathbf{M}_{\mathbf{R}}$ *if and only if* A *is the sum* $A_1 + \cdots + A_s$ *of Moore invertible matrices* A_i *in* $\mathbf{M}_{\mathbf{R}}$ *such that*

$$u(A_i)u(A_i)^{\dagger}u(A_j)u(A_j)^{\dagger} = 0 \qquad \textit{for} \quad i \neq j.$$

In this case, $A^{\dagger} = G(A_1) + \cdots + G(A_s)$.

Proof. First, assume that $A = A_1 + \cdots + A_s$, where for every i, $A_i^{\dagger} = G(A_i)$, and where the associated Moore idempotents $e(A_i) = u(A_i)u(A_i)^{\dagger}$ are pairwise orthogonal in the sense that the product of distinct pairs is zero. Since $e(A_i)A_i = A_i$, $e(A_i)G(A_i) = G(A_i)$, and $e(A_i)e(A_j) = 0$ for $i \neq j$, then $A^{\dagger} = A_1^{\dagger} + \cdots + A_s^{\dagger} = G(A_1) + \cdots + G(A_s)$.

Conversely, $A = 0$ is clearly Moore invertible. If $A \neq 0$ in $\mathbf{M}_{\mathbf{R}}$ has a Moore-Penrose inverse, then we show that A is representable as a sum $A_1 + \cdots + A_s$ of Moore invertible matrices with pairwise orthogonal Moore idempotents such that rank A = rank $A_1 > \cdots >$ rank $A_s > 0$.

The proof is by induction on r. If $r = 1$, then the conclusion follows from Corollary 2. Specifically, $s = 1$, and $A = A_1 = u(A)u(A)^{\dagger}A$ is Moore invertible.

Thus, suppose that the conclusion is valid for Moore-Penrose invertible matrices of determinantal rank less than r, and let A in $\mathbf{M}_{\mathbf{R}}$ be of determinantal rank r with Moore-Penrose inverse $A^{\dagger}$. By Corollary 2, $u(A)^{\dagger}$ exists. Let $e(A) = u(A)u(A)^{\dagger}$. If $A = e(A)A$, then the conclusion holds with $s = 1$ and $A_1 = e(A)A$. Otherwise, consider $A = e(A)A + [1 - e(A)]A$. Since $1 - e(A)$ is a symmetric idempotent, then $\{[1 - e(A)]A\}^{\dagger} = [1 -$

$e(A)]A^{\dagger}$. Also, by the properties of the r-compound and by Corollary 2,

$$C_r\{[1 - e(A)]A\} = [1 - e(A)]^r C_r(A)$$

$$= [1 - e(A)]C_r(A) = [1 - e(A)]e(A)C_r(A) = 0.$$

That is, $[1 - e(A)]A \neq 0$ is of determinantal rank less than r and has a Moore-Penrose inverse. By the induction hypothesis,

$$[1 - e(A)]A = A_2 + \cdots + A_s,$$

where each A_i is Moore invertible with pairwise orthogonal Moore idempotents $e(A_i)$, and

$$r > \operatorname{rank}\{[1 - e(A)]A\} = \operatorname{rank} A_2 > \cdots > \operatorname{rank} A_s > 0.$$

Since $A_i = e(A_i)[1 - e(A)]A$, then $u(A_i)$, and hence $e(A_i)$, is divisible by $1 - e(A)$. In particular, $e(e(A)A)e(A_i) = e(A)e(A_i) = 0$. Consequently, $A = e(A)A + A_2 + \cdots + A_s$, where each summand is Moore invertible, the associated Moore idempotents are pairwise orthogonal, and

$$r = \operatorname{rank}[e(A)A] > \operatorname{rank} A_2 > \cdots > \operatorname{rank} A_s > 0.$$

This completes the induction. ■

COROLLARY. *If A in $\mathbf{M_R}$ has a Moore-Penrose inverse, then A is uniquely representable as a sum $A_1 + \cdots + A_s$ of Moore invertible matrices with pairwise orthogonal Moore idempotents such that* rank $A_1 > \cdots >$ rank A_s. *In this case,* rank A = rank A_1.

Proof. If $A^{\dagger}$ exists, then by the proof of the theorem, such a representation exists. We now show that there is at most one such representation. Specifically, let $A_1 + \cdots + A_s$ and $B_1 + \cdots + B_t$ be such representations of $A : m \to n$ of rank r. Without loss of generality let $s \leqslant t$.

First, if $B_t = 0$, since $C_0(0) = (1)$, then $e(B_t) = 1$, $e(B_i) = 0$ for $i \neq t$, and $B_i = e(B_i)A = 0$ for every i; that is, $A = 0$, $s = 1 = t$, and $A_1 = B_1$.

Next, suppose that $B_t \neq 0$. Since $A_i = e(A_i)A_i$, by orthogonality of the Moore idempotents $e(A_i)$, we have $A_1 = e(A_1)A$ and rank $A_1 \leqslant r$. If $\alpha \in G_{r,m}$ and $\beta \in G_{r,n}$, then, by orthogonality and the fact that rank $A_i < r$ whenever $i > 1$,

$$\det A_{\alpha\beta} = \det A_{1\alpha\beta} + \cdots + \det A_{s\alpha\beta} = \det A_{1\alpha\beta}.$$

In particular, rank $A_{1\alpha\beta} \geqslant r$. Consequently, rank $A_1 = r$, $u(A) = u(A_1)$, $e(A) = e(A_1)$, and $A_1 = e(A_1)A = e(A)A$. Likewise, $B_1 = e(A)A$; in particular, $A_1 = B_1$.

If $s > 1$, then a repeat of this same argument on $A - e(A)A = A_2 + \cdots + A_s = B_2 + \cdots + B_t$ provides $A_2 = B_2$. Indeed, further repeats give $A_1 = B_1, \dots, A_s = B_s$.

If $t > s$, then $0 = B_{s+1} + \cdots + B_t$, rank $B_{s+1} = 0$, $t = s + 1$, and $B_t = 0$, which is not the case. Consequently, $s = t$ and the representations are identical. ■

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